Bounding Multivariate Trigonometric Polynomials

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***Abstract*—The extremal values of multivariate trigonometric polynomials are of interest in fields ranging from control theory to filter design, but finding the extremal values of such a polynomial is generally NP-Hard. In this paper, we develop simple and efficiently computable estimates of the extremal values of a multivariate trigonometric polynomial directly from its samples. We provide an upper bound on the modulus of a complex trigonometric polynomial, and develop upper and lower bounds for real trigonometric polynomials. For a univariate polynomial, these bounds are tighter than existing bounds, and the extension to multivariate polynomials is new. As an application, the lower bound provides a sufficient condition to certify global positivity of a real trigonometric polynomial.**

I. Introduction

A. Motivation

Trigonometric polynomials are intimately linked to discrete-time signal processing, arising in problems of controls, communications, filter design, and super resolution, among others. For example, the Discrete-Time Fourier Transform (DTFT) converts a sequence of length n into a trigonometric polynomial of degree $n-1$. Multivariate trigonometric polynomials arise in a similar fashion, as the d-dimensional DTFT yields a d-variate trigonometric polynomial.

The extremal values of a trigonometric polynomial are often of interest. In an Orthogonal Frequency Division Multiplexing (OFDM) communication system, the transmitted signal is a univariate trigonometric polynomial, and the maximum modulus of this signal must be accounted for when designing power amplifiers [1]. The maximum modulus of a trigonometric polynomial is related to the stability of a control system in the face of perturbations [2]. The maximum gain and attenuation of a Finite Impulse Response (FIR) filter are the maximum and minimum values of a real and non-negative trigonometric polynomial. Unfortunately, determining the extremal values of a multivariate polynomial given its coefficients is NP-Hard [3], [4].

An approximation to the extremal values can be found by discretizing the polynomial and performing a grid search, but this method is sensitive to the discretization level. Instead, one can try to find the extremal values using an optimization-based approach. However, iterative descent algorithms are prone to finding local optima as a generic polynomial is not a convex function. The sum-of-squares machinery provides an alternative approach: extremal values of a polynomial can be found by solving a hierarchy of semidefinite program (SDP) feasibility problems [2], [4], [5]. Truncating the sequence of SDPs provides a lower (or upper) bound to the minimum (or maximum) of the polynomial. However, the size of the SDPs grows exponentially in the number of variables, d, and polynomially in the degree, n, limiting the applicability of this approach.

In many applications we have access to samples of the polynomial rather than to the coefficients of the polynomial itself. Equally spaced samples of a trigonometric polynomial arise, for instance, when computing the Discrete Fourier Transform (DFT) of a sequence. Given enough samples, the polynomial can be evaluated at any point by periodic interpolation, and thus grid search or optimization-based approaches can still be used; however, the previously described issues of discretization error, local minima, and complexity remain.

In this paper, we derive simple estimates for the extremal values of a multivariate trigonometric polynomial directly from its samples, *i.e.* with no interpolation step. For a complex polynomial we provide an upper bound on its modulus, while for a real trigonometric polynomial we provide upper and lower bounds. Upper bounds of this style have been derived for univariate trigonometric polynomials– our work provides an extension to the multivariate case. We describe two sample applications that benefit from our lower bound and from the extension to multivariate polynomials.

i) **Design of Perfect Reconstruction Filter Banks**. A multi-rate filter bank in d dimensions is characterized by its polyphase matrix, $H(z) \in \mathbb{C}^{m \times n}$, where each entry in the matrix is a d-variate Laurent polynomial[1] in $z \in \mathbb{C}^d$ [6].

Many important properties of the filter bank can be inferred from the polyphase matrix. A filter bank is said to be *perfect reconstruction* (PR) if any signal can be recovered, up to scaling and a shift, from its filtered form. The design and characterization of multirate filter banks in one dimension is well understood, but becomes difficult in higher dimensions due to the lack of a spectral factorization theorem [7]–[11]. The perfect reconstruction condition is equivalent to the strict positivity of the real trigonometric polynomial $p_H(\omega) = \det\left(H^*(e^{j\omega})H(e^{j\omega})\right)$ [6], [12]. The lower bounds developed in this paper provide a sufficient condition to verify the perfect reconstruction property from samples of $p_H(\omega)$ which are easily obtained using the DFT.

ii) **Estimating the smallest eigenvalue of a Hermitian Block Toeplitz matrix with Toeplitz Blocks.**

Toeplitz matrices describe shift-invariant phenomena and are found in countless applications. Toeplitz matrices model convolution with a finite impulse response filter, and the covariance matrix formed from a random vector drawn from a wide-sense stationary (WSS) random process is symmetric

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[1] A Laurent polynomial allows negative powers of the argument.

and Toeplitz. An $n \times n$ Toeplitz matrix is of the form

$$X_n = \begin{bmatrix} x_0 & x_{-1} & x_{-2} & \cdots & x_{-n+1} \\ x_1 & x_0 & x_{-1} & & \\ x_2 & x_1 & x_0 & & \vdots \\ \vdots & & & \ddots & \\ x_{n-1} & & & \cdots & x_0 \end{bmatrix}, \tag{1}$$

and a Hermitian symmetric Toeplitz matrix satisfies $x_i^* = x_{-i}$. Associated with X_n is the trigonometric polynomial [2]

$$\hat{x}(\omega) = \sum_{k=-n}^{n} x_k e^{j\omega k}, \qquad -\pi \le \omega < \pi, \tag{2}$$

with coefficients

$$x_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\omega) e^{-jk\omega} dt, \qquad k \in \mathbb{Z}. \tag{3}$$

The polynomial $\hat{x}$ is known as the *symbol* of X_n. If the symbol is real then X_n is Hermitian, and if $\hat{x}$ is strictly positive then X_n is positive definite.

A vast array of literature has examined the connections between a real symbol $\hat{x}$ and the eigenvalues of the Hermitian Toeplitz matrices X_n as $n \to \infty$; see [13], [14] and references therein. One result of particular interest states that the eigenvalues of X_n are upper and lower bounded by the supremum and infimum of the symbol.

The smallest eigenvalue of a Toeplitz matrix is of interest in many applications [15]–[17], and there are several iterative algorithms to efficiently calculate this eigenvalue [18]. We propose a non-iterative estimate of the smallest and largest eigenvalues of X_n by first bounding the eigenvalues in terms of the symbol, then bounding the symbol in terms of the entries of X_n.

Shift invariant phenomena in two dimensions are described by Block Toeplitz matrices with Toeplitz Blocks (BTTB). The symbol for a BTTB matrix is a bi-variate trigonometric polynomial, and the bounds developed in this paper hold in this case.

B. Notation

For a set $\mathbb{X}$, let $\mathbb{X}^d$ be the d-fold Cartesian product $\mathbb{X} \times \ldots \times \mathbb{X}$. Let $\mathbb{T} = [0, 2\pi]$ be the torus and $\mathbb{Z}$ be the integers. The set $\{0, \ldots N-1\}$ is written $[N]$. We denote the space of d-variate trigonometric polynomials with maximum component degree n as

$$T_n^d \triangleq \text{span}\left\{e^{jk\cdot\omega} : \omega \in \mathbb{T}^d, k \in \mathbb{Z}^d, \|k\|_\infty \le n\right\}, \tag{4}$$

where $x \cdot y \triangleq \sum_{i=1}^d x_i y_i$ is the Euclidean inner product and $\|k\|_\infty = \max_{1\le i\le d} |k_i|$. An element of T_n^d is explicitly given by

$$p(\omega) = \sum_{k_1=-n}^{n} \cdots \sum_{k_d=-n}^{n} c_{k_1\ldots k_d} e^{jk_1\omega_1} \ldots e^{jk_d\omega_d}. \tag{5}$$

[2]This differs from the usual approach of describing Toeplitz matrices, wherein a Toeplitz matrix of size n is generated according to (3) for an underlying symbol and the behavior as $n \to \infty$ is investigated. Here, we work with a Toeplitz matrix of fixed size.

If the coefficients satisfy $c_{k_1,\ldots,k_d} = c^*_{-k_1,\ldots,-k_d}$, then $p(\omega)$ is real for all ω and p is said to be a *real trigonometric polynomial*. We denote the space of real trigonometric polynomials by $\bar{T}_n^d$. For $p \in T_n^d$ let $\|p\|_\infty = \max_{\omega\in\mathbb{T}^d} |p(\omega)|$. We write the set of N equidistant sampling points on $\mathbb{T}$ as

$$\Theta_N \triangleq \left\{\omega_k = k\frac{2\pi}{N} : k = 0, \ldots, N-1\right\}, \tag{6}$$

and on $\mathbb{T}^d$ as Θ_N^d, given by the d-fold Cartesian product $\Theta_N \times \ldots \times \Theta_N$. The maximum modulus of p over Θ_N^d is

$$\|p\|_{N^d,\infty} \triangleq \max_{\omega\in\Theta_N^d} |p(\omega)|. \tag{7}$$

C. Problem Statement and Existing Results

Let $p \in \bar{T}_n^d$. Our goal is to find scalars $a \le b$, depending only on N, d, and the N^d samples $\{p(\omega) : \omega \in \Theta_N^d\}$, such that

$$a \le p(\omega) \le b. \tag{8}$$

For complex trigonometric polynomials, $p \in T_n^d$, we want an upper bound on the modulus; a lower bound on the modulus can be obtained by considering the real trigonometric polynomial $p' \in \bar{T}_n^{2d} : \omega \mapsto |p(\omega)|^2$.

By the periodic sampling theorem (Lemma 1), trigonometric interpolation perfectly recovers $p \in T_n^d$ from $(2n+1)^d$ uniformly spaced samples. A standard result of approximation theory states [19], [20]

$$\|p\|_\infty \le \|p\|_{(2n+1)^d,\infty} \left(\frac{\pi+4}{\pi} + \frac{2}{\pi}\log(2n+1)\right)^d, \tag{9}$$

but this becomes weak as the polynomial degree n or the dimension d of its domain increases. A more stable estimate is obtained by using non-uniformly spaced samples. However, in many applications the sampled polynomial is obtained using the DFT, thus providing uniformly spaced samples.

Our aim is to get stronger estimates by using more (uniformly spaced) samples than are required by the periodic sampling theorem. Upper bounds for *univariate* trigonometric polynomials have been developed using this strategy. Let $p \in T_n$. Given an integer m and $N = 2m > 2n+1$ samples of p, Ehlich and Zeller showed

$$\|p\|_\infty \le \left(\cos\left(\frac{\pi n}{2m}\right)\right)^{-1} \|p\|_{N,\infty} \tag{10}$$

and this bound is sharp if n is a divisor of m.

Wunder and Boche developed a more flexible bound: given $N \ge 2n+1$, they showed [21]

$$\|p\|_\infty \le \sqrt{\frac{N+2n+1}{N-(2n+1)}}\|p\|_{N,\infty}. \tag{11}$$

Zimmermann *et al.* refined this bound to

$$\|p\|_\infty \le \frac{\|p\|_{N,\infty}}{\sqrt{1-\alpha}}, \tag{12}$$

where $\alpha = 2n/N$. The quantity α^{-1} is almost equal to the oversampling factor $\frac{N}{2n+1}$, and plays the same role: α is a decreasing function of N, and for $N \ge 2n+1$, we have $\alpha < 1$.

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The bounds (9) to (12) each have the form:

$$\|p\|_\infty \le C^d_{N,n} \|p\|_{N^d,\infty}, \tag{13}$$

where $C^d_{N,n}$ is a real, non-negative constant that depends on N, n and, in the case of (9), d. In the univariate case, Zimmermann *et al*. studied the optimal value of $C_{N,n}$ and showed that it depends only on N/n. They also characterized *extremal* polynomials, for which (13) holds with equality, and discussed a Remez-like algorithm to construct such polynomials for given N and n [1].

D. Contributions

Our contributions can be summarized as follows: (i) we develop upper bounds of the form (13) for *multivariate* trigonometric polynomials; these include both a multivariate extension of the bound (12), as well as a tighter bound for the case of low oversampling ($N \approx 2n+1$); (ii) we specialize and strengthen the bounds for real polynomials; and (iii) we derive a lower bound for real trigonometric polynomials.

II. Statement of Main Results

In this section we collect our main results; proofs are deferred to Sections III and IV. For simplicity we work with T^d_n, but the results can be easily strengthened by allowing for the component degree to vary in each of the d dimensions.

Our first task is to obtain bounds of the form (13) for multivariate trigonometric polynomials. We have a pair of such bounds:

Theorem 1. *Let $p \in T^d_n$. Take $N \ge 2n+1$ and set $\alpha = 2n/N$. Then*

$$\|p\|_\infty \le C^d_{N,n} \|p\|_{N^d,\infty}, \tag{14}$$

where

$$C^d_{N,n} \triangleq \frac{\left(\sup_{\omega\in\mathbb{T}} \left\{ \sum_{\omega_k\in\Theta_N} \left| \frac{\sin\left(\frac{N\omega}{2}\right) \sin\left(\frac{N-2n}{2}(\omega-\omega_k)\right)}{\sin^2((\omega-\omega_k)/2)} \right| \right\} \right)^d}{N^d(N-2n)^d} \tag{15}$$

$$\le (1-\alpha)^{-\frac{d}{2}}. \tag{16}$$

Further,

$$C^d_{N,n}\|p\|_{N^d,\infty} - \|p\|_\infty \le \left(\frac{dn}{N} + \mathcal{O}((dn/N)^2)\right)\|p\|_\infty. \tag{17}$$

The bound (15) involves only a univariate function and can be calculated numerically. Still, the expression is unwieldy; (16) is a simpler, but weaker, alternative.

We plot the behavior of $C_{N,n}$, given by (15) and (16) for the $d=1$ univariate case, in Fig. 1. Also shown in Fig. 1 are the optimal values of $C_{N,n}$ for integer oversampling factors, given by (10), and the values obtained using Zimmermann's Remez-like algorithm [1].

The upper bound (14) with $C^d_{N,n}$ given by (15) is nearly tight for $N/(2n) < 2$, whereas replacing $C^d_{N,n}$ by its upper bound (16) results in a weakening of (14) in this regime. This gap makes (15) particularly attractive in the d-variate case, where the bounds are raised to the d-th power, further increasing the gap between (15) and (16).

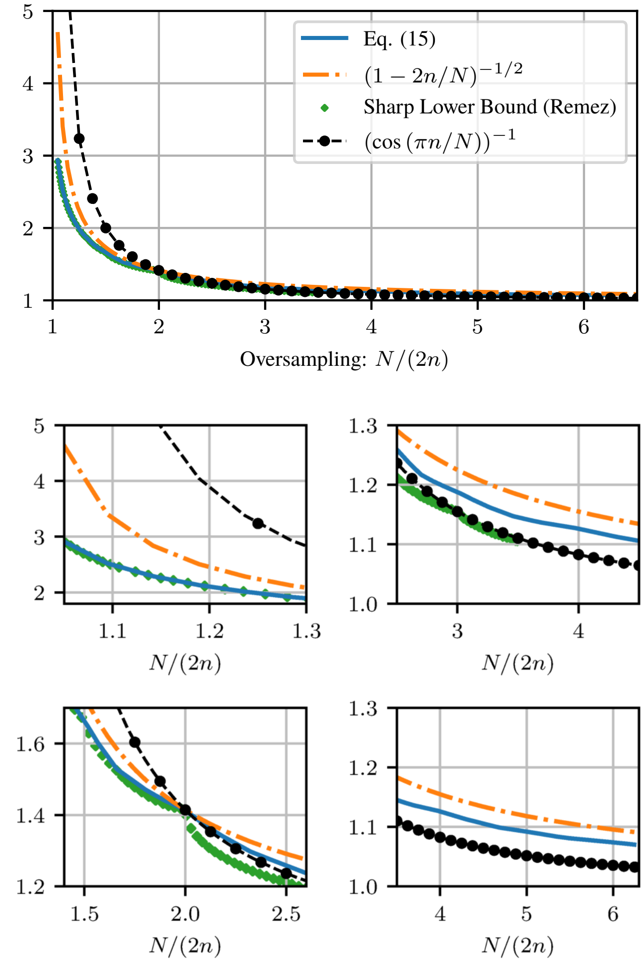


Fig. 1: Comparing upper bounds of the form (14) as a function of oversampling ratio, $N/2n$, calculated with $n=8$. Green diamonds indicate the optimal upper bound as calculated using a Remez-type algorithm [1, Fig. 2]. Black dots denote the upper bound (10) at valid locations, *i.e.* $N = 2m > 2n+1$.

However, for oversampling factor greater than two, *i.e.* $N/(2n) > 2$, the difference in using (15) or (16) becomes negligible. Both bounds coincide with the optimal value at $N = 4n$, and are within roughly 10% of the optimal value for large oversampling factors. Hence, both (15) and (16) are useful, in different oversampling regimes.

Next, we obtain a tighter estimate and a lower bound by restricting our attention to real polynomials.

Corollary 1. *Let $p \in \bar{T}^d_n$ and take $N \ge 2n+1$. Set $A \triangleq \max_{\omega\in\Theta^d_N} p(\omega)$, $B \triangleq \min_{\omega\in\Theta^d_N} p(\omega)$ and take $C^d_{N,n}$ as in*

Theorem 1. Then,

$$p(\omega) \leq \frac{1}{2}\left(A + B + C_{N,n}^d\left(A - B\right)\right), \tag{18}$$

$$p(\omega) \geq \frac{1}{2}\left(A + B - C_{N,n}^d\left(A - B\right)\right), \tag{19}$$

$$\|p\|_\infty \leq \frac{1}{2}\left(|A + B| + C_{N,n}^d(A - B)\right). \tag{20}$$

The estimates (18) and (20) coincide with (14) in the case that $\min_{\omega\in\Theta_N^d} p(\omega) = -\max_{\omega\in\Theta_N^d} p(\omega)$, and are tighter otherwise, making this refinement especially useful for non-negative polynomials.

By Theorem 1, $C_{N,n}^d \to 1$ as $N \to \infty$. Thus as $N \to \infty$, the right hand side of (19) approaches B, and by continuity we have $B = \min_{\omega\in\Theta_N^d} p(\omega) \to \min_{\omega\in\mathbb{T}^d} p(\omega)$. Thus the bound is tight as $N \to \infty$. In the case of $A = B$, the right hand side of (19) is $A = \|p\|_{N^d,\infty}$, and thus $p(\omega) > 0$ so long as the samples of p are not uniformly zero. This is expected, as otherwise the polynomial $p(\omega) - \|p\|_{N^d,\infty} \in T_n^d$ would vanish on a set of $N^d > (2n+1)^d$ points, which is impossible unless the polynomial is identically zero.

A little algebra on (19) establishes a sufficient condition to verify the strict positivity of a multivariate trigonometric polynomial.

Corollary 2. *Let $p \in \bar{T}_n^d$ and $N \geq 2n + 1$. Set $\alpha = 2n/N$. If $p(\omega) > 0$ for all $\omega \in \Theta_N^d$ and*

$$\kappa_N \triangleq \frac{\max_{\omega\in\Theta_N^d} p(\omega)}{\min_{\omega\in\Theta_N^d} p(\omega)} < \frac{C_{N,n}^d + 1}{C_{N,n}^d - 1} \tag{21}$$

then $p(\omega) > 0$ for all $\omega \in \mathbb{T}^d$. Furthermore, as $C_{N,n}^d \leq (1-\alpha)^{-\frac{d}{2}}$, (21) can be replaced by the more stringent, but easier to evaluate, condition

$$\kappa_N < \frac{1 + (1-\alpha)^{\frac{d}{2}}}{1 - (1-\alpha)^{\frac{d}{2}}}. \tag{22}$$

For $p \in \bar{T}_n^d$ with non-negative samples, we call the quantity κ_N in (21) the *N-sample dynamic range*.

Corollary 2 provides an easy way to certify strict positivity of a real, non-negative polynomial from its samples: simply calculate the dynamic range κ_N and verify that (21) or (22) holds. These conditions are easier to satisfy (as a function of the oversampling rate) for polynomials whose maximum and minimum sampled values are close to one another. Intuitively, if the sampled values of a real trigonometric polynomial are strictly positive and don't vary "too much", then the polynomial is strictly positive over its entire domain. For fixed n and d, the right hand side of (22) is an increasing function of N, illustrating a tradeoff: polynomials with a large amount of variation, and thus large values of κ_N, require larger oversampling factors N for the bounds to hold. Note that κ_N is not necessarily a monotone function of N, but is monotone in k when choosing $N = 2^k$. Fig. 2 illustrates the regions for which (21) and (22) hold.

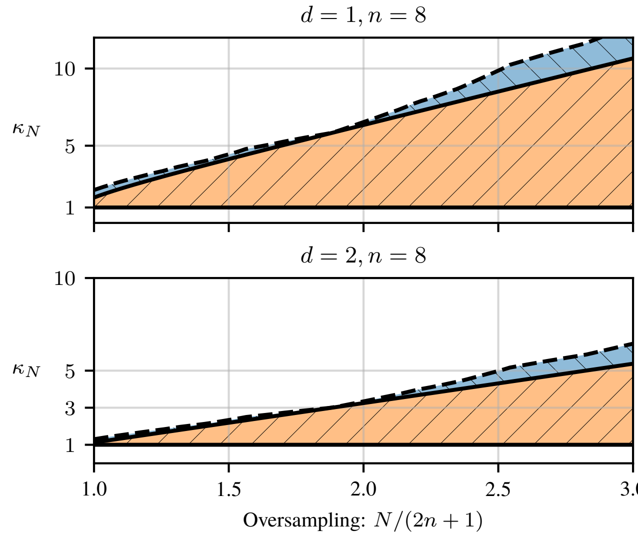


Fig. 2: Any $p \in \bar{T}_n^d$ with positive samples and whose N-sample signed dynamic range κ_N lies in the shaded region must be strictly positive. The orange shaded region is certified using (22), while the blue region uses (21).

III. Proof of Theorem 1

We begin by proving Theorem 1, which extends the upper bound (12) from univariate to multivariate polynomials and provides a tighter result for the case of low oversampling. Due to the separable nature of T_n^d (*e.g.* T_n^d is the d-fold tensor product of T_n with itself), the proof is similar to the univariate case [1]. We consider both real and complex trigonometric polynomials.

A. Interpolation by the Dirichlet Kernel

For $\mathbf{n} = [n_1, \ldots n_d] \in [N]^d$, the $\mathbf{n}$-th order Dirichlet kernel is the tensor product of d kernels, each of order n_i:

$$D_{\mathbf{n}}^d(\omega) \triangleq \sum_{|k_i|\leq n_i} e^{jk\cdot\omega} = \prod_{i=1}^d \frac{\sin\frac{2n_i+1}{2}\omega_i}{\sin\frac{\omega_i}{2}} \quad \omega \in \mathbb{T}^d, k \in \mathbb{Z}^d. \tag{23}$$

If $\mathbf{n}$ is identical in each index (*i.e.* $n_i = n$ for each $i \in [d]$) we write the kernel as $D_n^d(\omega)$. The Dirichlet kernel is key to the periodic sampling formula:

Lemma 1. *Let $p \in T_n^d$ be sampled on Θ_N^d. Let m be an integer with $m \geq n$. If $N > n + m$, then*

$$p(\omega) = \frac{1}{N^d} \sum_{\omega_k\in\Theta_N^d} p(\omega_k) D_m^d(\omega - \omega_k) \tag{24}$$

for all $\omega \in \mathbb{T}^d$.

Lemma 1 (*e.g.*, [22]) is the periodic counterpart of sinc interpolation arising in the Whittaker-Shannon interpolation formula. The bound (9) can be obtained from (24) when $N = 2n + 1$ [20].

B. Interpolation by the de la Vallée-Poussin Kernel

A better result is obtained by oversampling $(N > 2n + 1)$ and exploiting the nice properties of summation kernels.

Let n, m be integers with $m > n$ and define $\mathbb{V}^d_{n,m} = \{l \in \mathbb{Z}^d : n \leq l_i < m\}$. The n, m-th de la Vallée-Poussin kernel is defined as the moving average of Dirichlet kernels:

$$D^d_{n,m}(\omega) \triangleq \frac{1}{(m-n)^d} \sum_{\mathbf{n} \in \mathbb{V}^d_{n,m}} D^d_{\mathbf{n}}(\omega) \tag{25}$$

$$= \frac{1}{(m-n)^d} \prod_{i=1}^{d} \frac{\sin\left(\frac{m+n}{2}\omega_i\right) \sin\left(\frac{m-n}{2}\omega_i\right)}{\sin^2(\omega_i/2)}. \tag{26}$$

Taking $n = 0$ recovers the well-known Fejér kernel [23],

$$D^d_{0,m} = \frac{1}{m^d} \prod_{i=1}^{d} \frac{\sin^2\left(\frac{m}{2}\omega_i\right)}{\sin^2(\omega_i/2)}. \tag{27}$$

The Fejér kernel is used to derive the bound (11) [21].

Importantly, the de la Vallée-Poussin kernel inherits the reproducing property of the Dirichlet kernel.

Lemma 2. *For any $p \in T^d_n$ we have*

$$p(\omega) = \frac{1}{N^d} \sum_{\omega_k \in \Theta^d_N} p(\omega_k) D^d_{n,m}(\omega - \omega_k) \tag{28}$$

for all $\omega \in \mathbb{T}^d$ whenever $m > n$ and $N \geq n + m$.

Proof. Expanding the de la Vallée-Poussin kernel into a sum of Dirichlet kernels and applying Lemma 1,

$$\frac{1}{N^d} \sum_{\omega_k \in \Theta^d_N} p(\omega_k) D^d_{n,m}(\omega - \omega_k) \tag{29}$$

$$= \frac{1}{(m-n)^d} \sum_{\mathbf{n} \in \mathbb{V}^d_{n,m}} \frac{1}{N^d} \sum_{\omega_k \in \Theta^d_N} p(\omega_k) D^d_{\mathbf{n}}(\omega - \omega_k) \tag{30}$$

$$= \frac{1}{(m-n)^d} \sum_{\mathbf{n} \in \mathbb{V}^d_{n,m}} p(\omega) = p(\omega). \tag{31}$$

□

C. Proof of Theorem 1

The upper bound of Theorem 1 depends on estimates of $\sum_{\omega_k \in \Theta^d_N} \left|D^d_{n,m}(\omega - \omega_k)\right|$, which we collect into a pair of lemmas.

Lemma 3. *Take $N \geq 2n + 1$. Then, for all $\omega \in \mathbb{T}^d$,*

$$\sum_{\omega_k \in \Theta^d_N} \left|D^d_{n,N-n}(\omega - \omega_k)\right|$$

$$\leq \left(\sup_{\omega \in \mathbb{T}} \sum_{\omega_k \in \Theta_N} \left|D_{n,N-n}(\omega - \omega_k)\right|\right)^d \tag{32}$$

$$= \frac{\left(\sup_{\omega \in \mathbb{T}} \left\{ \sum_{\omega_k \in \Theta_N} \left| \frac{\sin\left(\frac{N\omega}{2}\right) \sin\left(\frac{N-2n}{2}(\omega - \omega_k)\right)}{\sin^2((\omega - \omega_k)/2)} \right| \right\}\right)^d}{(N-2n)^d}.$$

Proof. First, we fix notation: for $\omega_k \in \Theta^d_N$ and $k \in [N]^d$, we define $\omega_{k_i} = 2\pi k_i / N$. Using (26), we have

$$\sum_{\omega_k \in \Theta^d_N} \left|D^d_{n,N-n}(\omega - \omega_k)\right| (N - 2n)^d$$

$$= \sum_{\omega_k \in \Theta^d_N} \prod_{i=1}^{d} \left| \frac{\sin\left(\frac{N}{2}(\omega_i - \omega_{k_i})\right) \sin\left(\frac{N-2n}{2}(\omega_i - \omega_{k_i})\right)}{\sin^2((\omega_i - \omega_{k_i})/2)} \right|$$

$$\leq \left(\sup_{\omega \in \mathbb{T}} \sum_{\omega_k \in \Theta_N} \left| \frac{\sin\left(\frac{N}{2}(\omega - \omega_k)\right) \sin\left(\frac{N-2n}{2}(\omega - \omega_k)\right)}{\sin^2((\omega - \omega_k)/2)} \right|\right)^d \tag{33}$$

$$= \left(\sup_{\omega \in \mathbb{T}} \sum_{\omega_k \in \Theta_N} \left| \frac{\sin\left(\frac{N\omega}{2}\right) \sin\left(\frac{N-2n}{2}(\omega - \omega_k)\right)}{\sin^2((\omega - \omega_k)/2)} \right|\right)^d,$$

where the final step follows from $\left|\sin\left(\frac{N}{2}(\omega - 2\pi k/N)\right)\right| = \left|\sin\left(\frac{N\omega}{2}\right)\right|$ for $k \in [N]$. The bound (32) is obtained by replacing (33) with the definition of $D_{n,N-n}(\omega)$ given by (26). □

The following lemma for univariate trigonometric polynomials is key to the derivation of (12). [3]

Lemma 4. *Let $m > n$ and take $N \geq n + m$. Then*

$$\sum_{\omega_k \in \Theta_N} \left|D_{n,m}(\omega - \omega_k)\right| \leq N \left(\frac{m+n}{m-n}\right)^{\frac{1}{2}} \tag{34}$$

for all $\omega \in \mathbb{T}$. In particular, taking $N \geq 2n+1$ and $m = N-n$ yields

$$\sum_{\omega_k \in \Theta_N} \left|D_{n,N-n}(\omega - \omega_k)\right| \leq N \left(\frac{N}{N-2n}\right)^{\frac{1}{2}}. \tag{35}$$

Proof. See [1, Theorem 1]. □

We are now set to complete the proof of Theorem 1.

Proof of Theorem 1. Without loss of generality, assume $\|p\|_{N^d,\infty} = 1$. Then, by Lemma 2, we have

$$|p(\omega)| = \left| \frac{1}{N^d} \sum_{\omega_k \in \Theta^d_N} p(\omega_k) D^d_{n,N-n}(\omega - \omega_k) \right| \tag{36}$$

$$\leq \frac{1}{N^d} \sum_{\omega_k \in \Theta^d_N} \left|p(\omega_k) D^d_{n,N-n}(\omega - \omega_k)\right| \tag{37}$$

$$\leq \frac{1}{N^d} \sum_{\omega_k \in \Theta^d_N} \left|D^d_{n,N-n}(\omega - \omega_k)\right| \tag{38}$$

where (37) and (38) follow from the triangle inquality and Hölder's inequality, respectively.

[3] A multivariate extension is straightforward, but not used in the proof of Theorem 1 and is omitted here.

Now, applying Lemma 3, we have

$$|p(\omega)| \leq N^{-d}\left(\sup_{\omega\in\mathbb{T}} \sum_{\omega_k\in\Theta_N} |D_{n,N-n}(\omega-\omega_k)|\right)^d \tag{39}$$

$$= \frac{\left(\sup_{\omega\in\mathbb{T}} \left\{ \sum_{\omega_k\in\Theta_N} \left| \frac{\sin\left(\frac{N\omega}{2}\right)\sin\left(\frac{N-2n}{2}(\omega-\omega_k)\right)}{\sin^2((\omega-\omega_k)/2)} \right| \right\}\right)^d}{N^d(N-2n)^d}, \tag{40}$$

which implies (14)-(15). Applying the bound (35) of Lemma 4 to (39) yields

$$|p(\omega)| \leq \left(\frac{N}{N-2n}\right)^{\frac{d}{2}} = (1-\alpha)^{-\frac{d}{2}}, \tag{41}$$

which establishes (16).

Finally, as $N \geq 2n+1$, by Taylor's theorem we have $(1-\alpha)^{-\frac{d}{2}} = 1 + \frac{dn}{N} + \mathcal{O}((dn/N)^2)$. It follows that

$$\begin{aligned} C^d_{N,n}\|p\|_{N^d,\infty} - \|p\|_\infty &\leq \left(C^d_{N,n} - 1\right)\|p\|_\infty \\ &\leq \left((1-\alpha)^{-\frac{d}{2}} - 1\right)\|p\|_\infty \\ &= \left(\frac{dn}{N} + \mathcal{O}((dn/N)^2)\right)\|p\|_\infty, \end{aligned}$$

where we have used $\|p\|_{N^d,\infty} \leq \|p\|_\infty$. □

IV. Proof of Refinement and Lower Bound For Real Trigonometric Polynomials

We now restrict our attention to real trigonometric polynomials. We will use the shorthand notation $A \triangleq \max_{\omega\in\Theta^d_N} p(\omega)$ and $B \triangleq \min_{\omega\in\Theta^d_N} p(\omega)$. Note both A and B are (not necessarily monotonic) functions of N.

The bound of Theorem 1 is at its tightest whenever the samples of $p(\omega)$ are centered about zero, *i.e.* $\min_{\omega\in\Theta^d_N} p(\omega) = -\max_{\omega\in\Theta^d_N} p(\omega)$, and can be loose otherwise. To see this, take $c > 0$ and consider the shifted polynomial $\tilde{p}(\omega) = p(\omega) + c$. Applying Theorem 1 yields

$$\begin{aligned} \|\tilde{p}\|_\infty &\leq C^d_{N,n}\|\tilde{p}\|_{N^d,\infty} & (42)\\ &\leq C^d_{N,n}(\|p\|_{N^d,\infty} + c). & (43) \end{aligned}$$

Applying the triangle inequality in advance of Theorem 1 results in

$$\|\tilde{p}\|_\infty \leq \|p\|_\infty + c \leq C^d_{N,n}\|p\|_{N^d,\infty} + c, \tag{44}$$

which may be much smaller than (43), but presupposes knowledge of c. While we do not know this offset, it can be estimated from the samples of $\tilde{p}$. This motivates our refined bound, Corollary 1, which we now prove.

Corollary 1. *Let $p \in \bar{T}^d_n$ and take $N \geq 2n+1$. Set $A \triangleq \max_{\omega\in\Theta^d_N} p(\omega)$, $B \triangleq \min_{\omega\in\Theta^d_N} p(\omega)$ and take $C^d_{N,n}$ as in Theorem 1. Then,*

$$\begin{aligned} p(\omega) &\leq \frac{1}{2}\left(A + B + C^d_{N,n}(A-B)\right), & (18)\\ p(\omega) &\geq \frac{1}{2}\left(A + B - C^d_{N,n}(A-B)\right), & (19)\\ \|p\|_\infty &\leq \frac{1}{2}\left(|A+B| + C^d_{N,n}(A-B)\right). & (20) \end{aligned}$$

Proof of Corollary 1. If $A = B$ then $p(\omega) - A$ vanishes on a set of $N^d \geq (2n+1)^d$ points; thus $p(\omega)$ is the constant polynomial $p(\omega) = A$ and (18) to (20) hold with equality.

Define $q \in T^d_n$ as $q(\omega) \triangleq p(\omega) - \frac{A+B}{2}$, which satisfies

$$\|q\|_{N^d,\infty} = \left|A - \frac{A+B}{2}\right| = \frac{A-B}{2}. \tag{45}$$

By Theorem 1, we have for all $\omega \in \mathbb{T}^d$,

$$|q(\omega)| \leq C^d_{N,n}\frac{A-B}{2}. \tag{46}$$

Combined with the definition of $q(\omega)$, we have

$$-C^d_{N,n}\frac{A-B}{2} \leq p(\omega) - \frac{A+B}{2} \leq C^d_{N,n}\frac{A-B}{2}, \tag{47}$$

and rearranging gives (18) and (19).

Finally, we have

$$\begin{aligned} |p(\omega)| &\leq |q(\omega)| + \left|\frac{A+B}{2}\right| & (48)\\ &\leq C^d_{N,n}\frac{A-B}{2} + \left|\frac{A+B}{2}\right|, & (49) \end{aligned}$$

yielding (20). □

V. Examples

A. Univariate Example

Fig. 3 illustrates our bounds for a randomly chosen univariate real trigonometric polynomial, $p \in \bar{T}^1_8$, given by[4]

$$\begin{aligned} p(\omega) \triangleq 3.9 + \frac{1}{2}\Big(&0.4\cos(\omega) + 1.0\sin(\omega) \\ &+ 2.2\cos(2\omega) + 1.9\sin(2\omega) - 1.0\cos(3\omega) + 1.0\sin(3\omega) \\ &- 0.2\cos(4\omega) - 0.1\sin(4\omega) + 0.4\cos(5\omega) + 0.1\sin(5\omega) \\ &+ 1.5\cos(6\omega) + 0.8\sin(6\omega) + 0.1\cos(7\omega) + 0.4\sin(7\omega) \\ &+ 0.3\cos(8\omega) + 1.5\sin(8\omega)\Big). \end{aligned} \tag{50}$$

Note that the bounds are not necessarily monotonic functions of N. We see that an oversampling factor of 1.3, or $N = 23$, is enough samples to certify the strict positivity of this polynomial.

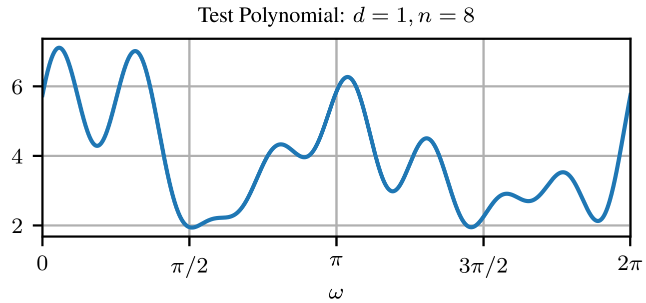
B. Trivariate Example

For simplicity, take $p \in \bar{T}^3_n$ to be $p(\omega) = D^3_n(\omega)/(2n+1)^3$, where D^3_n is the Dirichlet kernel (23) with (uniform) degree n and the scaling is such that $\|p\|_\infty = 1$.

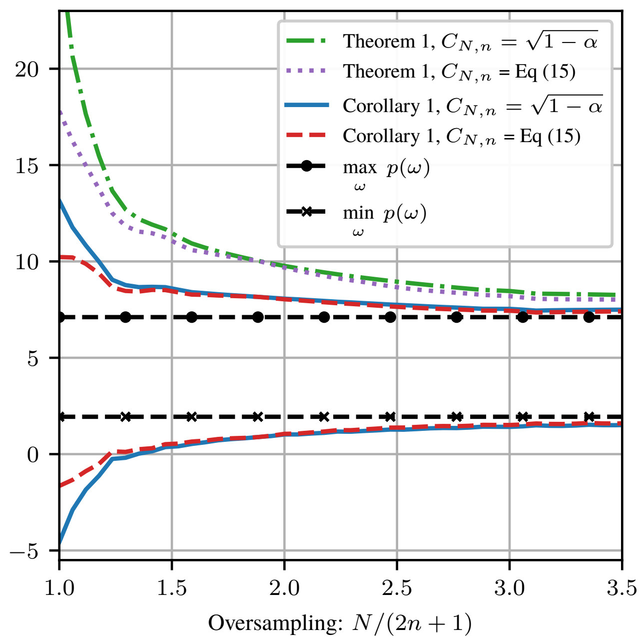
We obtain uniform samples of $p(\omega)$ over Θ^d_N by computing a zero-padded Discrete Fourier Transform. In particular, we embed an $n \times n \times n$ array of ones into an $N \times N \times N$ array of zeros, and apply the Fast Fourier Transform algorithm to this array. We choose N to be a favorable size for the FFT algorithm, such as a power of two. As we choose N proportional to the degree n of p, our method scales as $\mathcal{O}(n^d \log n)$ with $d = 3$ in this example.

Fig. 4 shows the estimates obtained using Corollary 1 as a function of N for a variety of orders n; the true maximum

[4]The coefficients were drawn from a standard normal distribution and rounded to the first decimal point.



(a)



(b)

Fig. 3: Example of upper and lower bounds for $p \in \bar{T}_8^1$ given by (50). (a): Test Polynomial. (b): Upper and lower bounds as a function of oversampling rate.

value of $p(\omega)$ is 1 and the minimum can be shown to be roughly $-2/(3\pi) \approx -0.22$. Evaluating the bounds for $n = 32$ and $N = 512$ took roughly one second on a workstation with an Intel i7-6700K CPU and 32GB of RAM.

To draw a comparison with the sum-of-squares framework, we use the `POS3POLY` MATLAB library, in particular the function `min_poly_value_multi_general_trig_3_5` [24]. This function finds the minimum value of a polynomial (given its coefficients) by a solving an SDP feasibility problem using an interior point method; the maximum value is obtained by calling the same function on $-p$. The per-iteration complexity of this method is $\mathcal{O}(n^{4d})$.

For $n = 7$, `POS3POLY` required 75 seconds to obtain the minimum value to within 3×10^{-3}; $n = 8$ required 260 seconds and found the minimum to within of 2×10^{-3}. The $n = 9$ case exhausted the system memory and was too large to solved on the workstation.

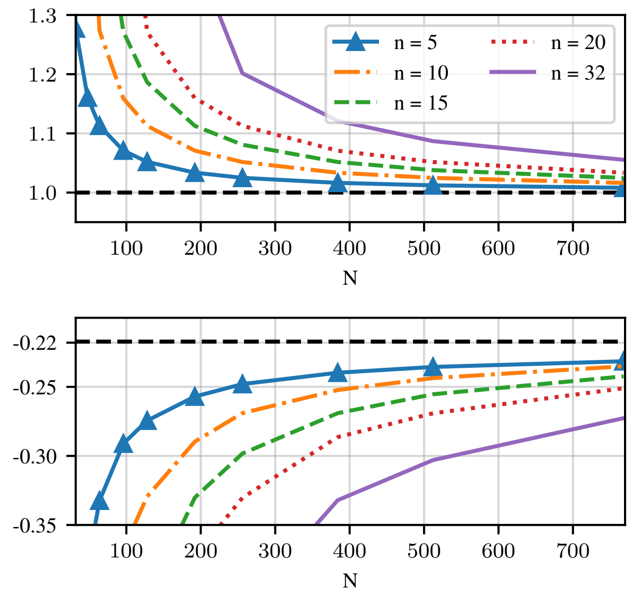


Fig. 4: Upper and lower bounds for the Dirichlet kernel of 3 variables using Corollary 1.

This is meant to be an illustrative, but certainly not exhaustive, comparison between the bounds presented in this paper and the sum-of-squares framework. Sum-of-squares methods are especially attractive if an exact solution is needed or if the polynomial has sparse coefficients, in which case the complexity can be dramatically reduced.

VI. Conclusion

We have proposed a fast and simple method to estimate the extremal values of a multivariate trigonometric polynomial directly from its samples. We have extended an existing upper bound from univariate to multivariate polynomials, and developed a strengthened upper bound and new lower bound for real trigonometric polynomials. The lower bound provides a new sufficient condition to certify global positivity of a real multivariate trigonometric polynomial. Future work will apply these results to the design of multidimensional perfect reconstruction filter banks.

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