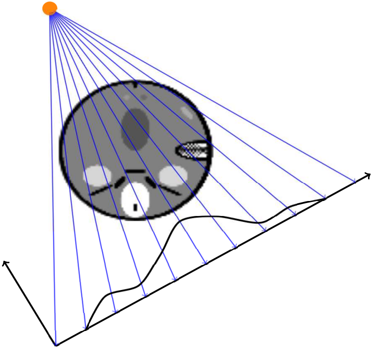
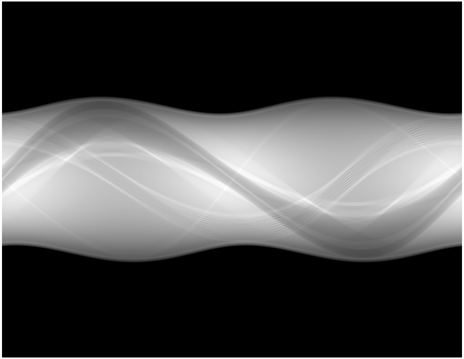
Tomographic Reconstruction with Adaptive Sparsifying Transforms

Luke Pfister & Yoram Bresler

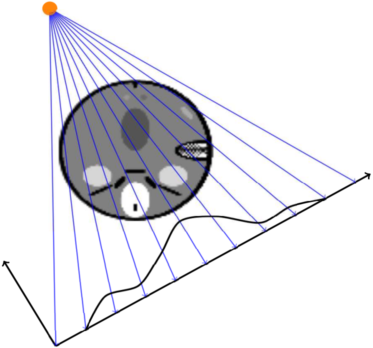
Computed Tomography

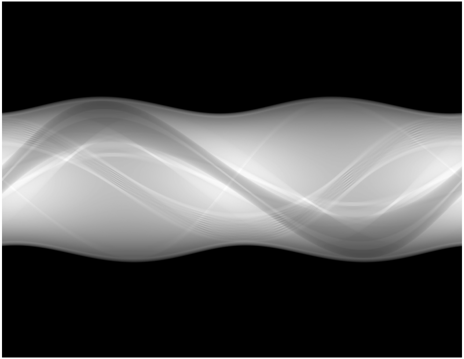




- Linear Measurements: $y = Ax$
- Reconstruction: Filtered Back Projection (FBP)

Computed Tomography





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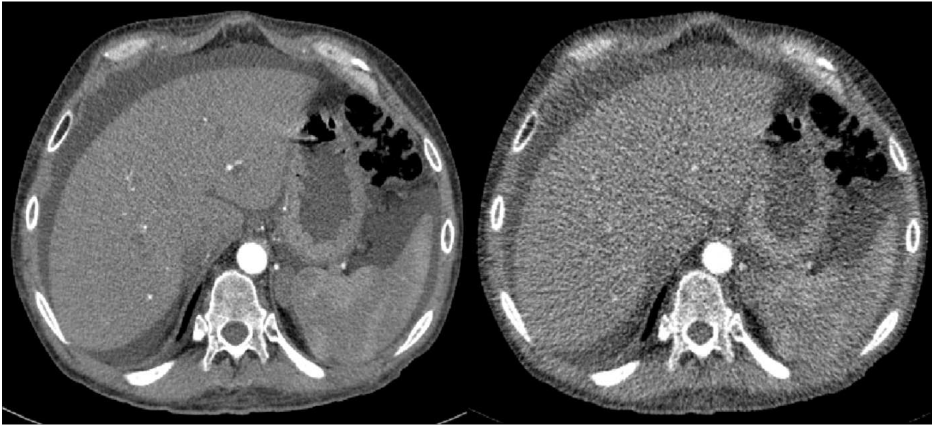
Low Dose Computed Tomography



Model-Based Image Reconstruction

- Three Ingredients
  - System Model
  - Noise Model
  - Signal Model
- Tie together into an optimization problem

Penalized Weighted Least-Squares

$$\min_x \frac{1}{2}\|y - Ax\|_W^2 + \lambda J(x)$$

Penalized Weighted Least-Squares

$$\min_x \frac{1}{2}\|y - Ax\|_W^2 + \lambda J(x)$$

- System Model
  - $y \in \mathbb{R}^M$: Log of CT data
  - $A \in \mathbb{R}^{M \times N}$: System matrix
  - $x \in \mathbb{R}^N$: Image estimate

Penalized Weighted Least-Squares

$$\min_x \frac{1}{2}\|y - Ax\|_W^2 + \lambda J(x)$$

- Noise Model
  - $W = diag\{w_i\}$
  - w_i are statistical weights
  - W is very poorly conditioned

Penalized Weighted Least-Squares

$$\min_x \frac{1}{2}\|y - Ax\|_W^2 + \lambda J(x)$$

- Signal Model
  - Regularizer $J(x) : \mathbb{R}^N \to \mathbb{R}$

Our Contribution

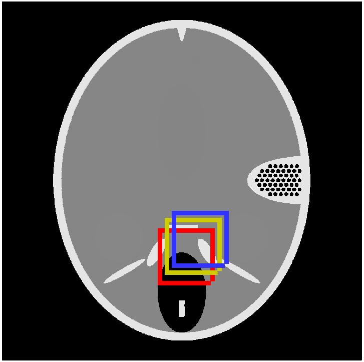
Propose fast, data-driven regularization with adaptive sparsifying transforms

Signal Models

Signal Models

- Better model $\implies$ better reconstruction
- Data-adative sparse representations: sparse signal models adapted for a **particular signal instance**
  - Usually patch based

Patch-based Signal Models

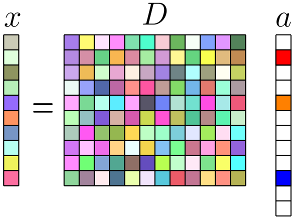


Sparse Signal Models

- Synthesis sparsity
- Transform sparsity

Synthesis Sparsity

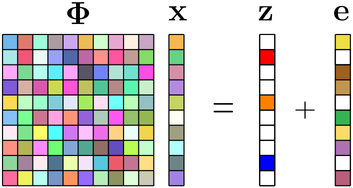
- $x = Da,\ a$ is sparse



- **Dictionary Learning**: Given $\{x_j\}_{j=1}^{P}$, find D and $\{a_j\}_{j=1}^{P}$
  - Applied to low-dose and limited-angle CT
  - Scales poorly with data size

Transform Sparsity

- $\Phi x = z + e, \quad z$ is sparse.
- e captures deviation from sparsity in transform domain



- **Transform Learning**: Given $\{x_j\}_{j=1}^{P}$, find Φ and $\{z_j\}_{j=1}^{P}$
  - Scales more gracefully with data size

Problem Formulation

Regularization with sparsifying transforms

$$J(x) = \min_{z,\Phi} \sum_j \frac{\lambda}{2} \|\Phi E_j x - z_j\|_2^2 + \gamma \|z_j\|_0 + \alpha(\|\Phi\|_F^2 - \log \det \Phi)$$

Regularization with sparsifying transforms

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Reconstruction Problem

$$\min_{x,\Phi,z_j} \frac{1}{2}\|y - Ax\|_W^2 + \lambda \sum_j \frac{1}{2}\|\Phi E_j x - z_j\|_2^2 + \lambda\gamma\|z_j\|_0$$
$$+ \lambda\alpha(\|\Phi\|_F^2 - \log\det\Phi))$$

Image Update

$$\min_{x} \frac{1}{2}\|y - Ax\|_W^2 + \frac{\lambda}{2}\sum_j \|\Phi E_j x - z_j\|_2^2$$

Transform Update

$$\min_{\Phi} \sum_j \|\Phi E_j x - z_j\|_2^2 + F(\Phi)$$

Sparse Code Update

$$\min_{z_j} \sum_j \|\Phi E_j x - z_j\|_2^2 + \gamma \|z_j\|_0$$

Regularizer Update

- Φ update

$$\Phi^{k+1} = \arg\min_{\Phi} \sum_j \frac{1}{2} \|\Phi E_j x - z_j\|_2^2 + \alpha \left(\|\Phi\|_F^2 - \log\det\Phi \right)$$

- Closed form solution! [Ravishankar, 2012]
- Requires three matrix products of size $p \times N$ by $N \times p$, and one SVD of size $p \times p$

Regularizer Update

- z_j update

$$z_j^{k+1} = \arg\min_{z_j} \frac{1}{2} \|\Phi E_j x - z_j\|_2^2 + \gamma \|z_j\|_0$$

- Closed-form solution using *hard thresholding*: $z_j^{k+1} = \mathcal{T}_\gamma \left(\Phi E_j x \right)$

$$\mathcal{T}_\gamma(a) = \begin{cases} 0, & |a| \leq \sqrt{\gamma} \\ a, & \text{else} \end{cases}$$

Image Update

$$\min_x \frac{1}{2}\|y - Ax\|_W^2 + \sum_j \frac{\lambda}{2}\|\Phi E_j x - z_j\|_2^2$$

- Weighted least-squares problem in x

$$H = A^T W A + \lambda \sum_j E_j^T \Phi^T \Phi E_j$$

Solution using ADMM [Ramani, 2012]

- Big Idea: Use variable splitting to untangle A and W

$$\min_x \frac{1}{2}\|y - v\|_W^2 + \sum_j \frac{\lambda}{2}\|\Phi E_j x - z_j\|_2^2$$

$$\text{subject to } \; v = Ax$$

Solution using ADMM [Ramani, 2012]

- Augmented Lagrangian

$$\mathcal{L}(x, v, \eta) = \frac{1}{2}\|y - v\|_W^2 + \sum_j \frac{\lambda}{2}\|\Phi E_j x - z_j\|_2^2 + \frac{\mu}{2}\|v - Ax - \eta\|_2^2 - \frac{\mu}{2}\|\eta\|_2^2$$

- Alternate between
  - Minimization over x
  - Minimization over v
  - Maximization over η

x-update

- Solve:

$$\left(\mu A^T A + \sum_j E_j^T \Phi^T \Phi E_j\right) x^{k+1} = \mu A^T (v^k - \eta^k) + \sum_j E_j^T \Phi^T z_j$$

- Linear unweighted least-squares in x
- Hessian $H = \mu A^T A + \sum_j E_j^T \Phi^T \Phi E_j$ is approximately shift-invariant
- Solve using Preconditioned Conjugate-Gradient (PCG) with circulant preconditioner

v-update

$$v^{k+1} = (W + \mu I)^{-1}(Wy + \mu(Ax^{k+1} + \eta^k))$$

η-update

$$\eta^{k+1} = \eta^k - v^{k+1} + Ax^{k+1}$$

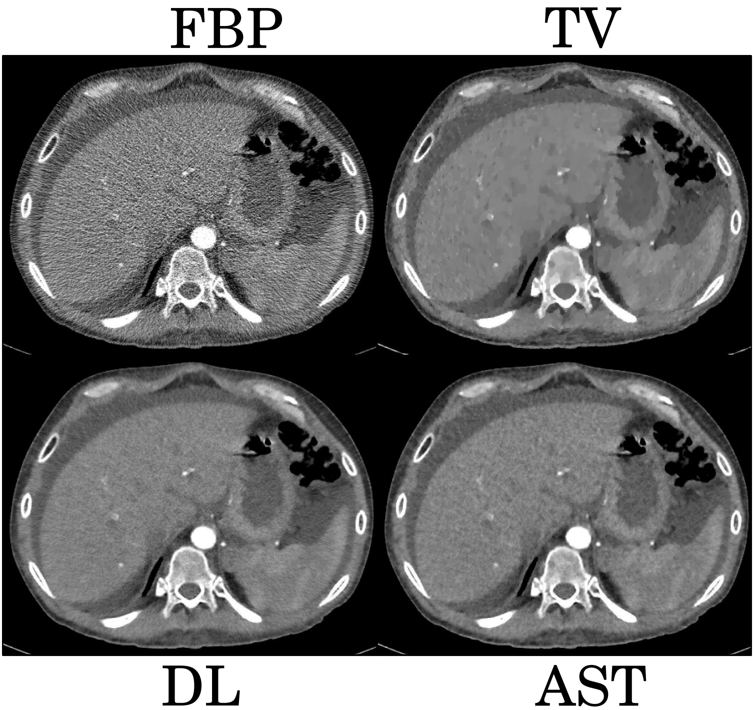
Overall Algorithm (AST-CT)

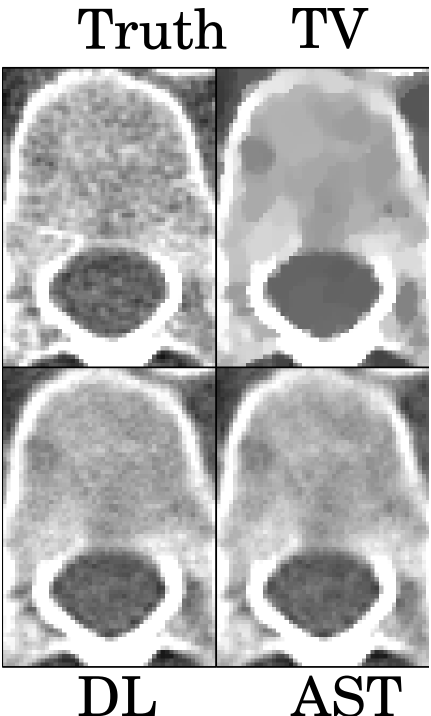
repeat
    repeat
        Update Φ
        $z_j^k \leftarrow \mathcal{T}_\gamma \Phi E_j x \; \forall j$
    until Halting condition
    $i \leftarrow 0$, $u^0 \leftarrow Ax^k$, $v^0 \leftarrow \vec{0}$
    repeat
        Use PCG to find approximate solution
        of $H\tilde{x}^{i+1} = \mu A^T(u^i - \eta^i) + \lambda \sum_j E_j^T \Phi^T z_j^i$
        $u^{i+1} \leftarrow (W + \mu I)^{-1} \left(Wy + \mu(A\tilde{x}^{i+1} + v^i)\right)$
        $\eta^{i+1} \leftarrow \eta^i - \left(u^{i+1} - A\tilde{x}^{i+1}\right)$
        $i \leftarrow i + 1$
    until Halting condition
    $x^{k+1} \leftarrow \tilde{x}^{i+1}$
until Halting Condidition

Experiments

Experiments

- Low-dose data synthesize from clinical image
- Total-variation (TV)
  - $J(x) = \|x\|_{TV}$
  - Apply variable splitting to data fidelity and regularizer
- Dictionary learning (DL):
  - $J(x) = \min_{D,a_j} \sum_j \|E_j x - D a_j\|_2^2 + \gamma \|a_j\|_0$
  - Solve with orthogonal matching pursuit and K-SVD


FBP
TV
DL
AST


Truth
TV
DL
AST

Experiments

	D/Φ Update	a/z Update	Image Update	Total
FBP	0	0	2.3	2.3
TV-CT	0	0	91.3	91.3
DL-CT	87.5	60.3	85.4	233.3
AST-CT	4.4	0.2	88.4	93.0

Conclusions

Conclusions

- Proposed the use of adaptive sparsifying transform regularization for low-dose CT reconstruction
- Performs as well as synthesis dictionary learning regularization at the speed of TV regularization

Thanks!